The structure of a hydromagnetic shock in steady plane motion

By G. S. S. LUDFORD
University of Maryland and Harvard University

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In this paper we discuss the structure of those oblique shocks which occur in plane hydromagnetic flow. We find that not all pairs of states satisfying the shock conditions can be linked by a one-dimensional flow of a viscous, electrically conducting fluid, while other pairs can be linked in more than one way. In the latter case a particular transition is singled out when further information is given concerning the three-dimensional problem of which the plane motion is an idealization.

Introduction

We consider the steady plane motion of an electrically conducting, perfect, viscous gas in the presence of a magnetic field in its own plane, on the basis of classical continuum theory. The local behaviour of the gas at a shock is studied by considering transitions in which all quantities, as functions of one co-ordinate x, change from finite initial values at $x = -\infty$ to finite final values at $x = +\infty$.

This problem was studied by Marshall (1955) for the particular case in which the x-component of the magnetic field (i.e. normal to the shock front) is zero throughout, so that the y-component of velocity may be taken zero also. He found that for a given initial state at $x = -\infty$ there is not only a unique final state at $x = +\infty$ but also a unique transition. In the general (oblique) case discussed in the present paper we find that there can be as many as three final states, with, moreover, an infinite number of transitions ending at one of them. The former indeterminacy is implied in several discussions of the necessary conditions which hold across a discontinuity surface in the flow of an inviscid, perfectly conducting fluid, and is presumably resolved in any particular problem by specifying further boundary conditions. However, such treatments can never discover whether two states satisfying these necessary conditions can in fact be joined by a transition solution, much less whether it is unique. Indeed, we shall also find that some pairs of states cannot be joined. In brief, the shock conditions are not sufficient to ensure that there is a unique transition joining any two states satisfying them [cf. the corresponding situation for the purely gas-dynamical shock as described by von Mises (1958)].

We consider in detail two particular cases, the first of which is Marshall's. The second corresponds simultaneously to switch-on (fast), switch-off (slow), and transverse shock discontinuities as identified by Friedrichs (1957). There is no transition solution for the last-mentioned and we must conclude that the transverse shock cannot occur in plane motion. Here we also encounter the lack of uniqueness mentioned in the last paragraph, which seems to point to the necessity of specifying how the current lines (lying along the z-direction) are closed at infinity, this constituting additional information about the three-dimensional problem of which the plane motion is an idealization. In fact, when we assume that the motion is the limit (at large radial distances) of an axially symmetric flow, for which the current lines are closed in the fluid (circles), the difficulty resolves itself.

Special interest lies in the behaviour of a transition solution in the limit of vanishing viscosity with non-zero resistivity. Marshall indicated that in his case a purely gas-dynamical discontinuity can appear (the Rankine–Hugoniot conditions are satisfied and the magnetic field is continuous), upstream of which there is an adjustment region to hydromagnetic values. We show explicitly that the adjustment may take place upstream or downstream, the former occurring in his case and for the switch-on shock, the latter for the switch-off shock.

Governing the motion are the Navier–Stokes equations, now containing terms arising from the Maxwell stresses and with the Joule heat appearing as a new source of dissipated energy, and Maxwell's equations supplemented by Ohm's law for a moving medium. The charge accumulation is neglected and, for simplicity, all material coefficients are assumed constant.

The argument is concerned with a phase plane, in which the transition solutions are represented by curves joining singular points of a certain system of differential equations. Any pair of singular points satisfies the shock conditions, and it is a question of deciding how many curves (if any) join the pair.

1. Equations of motion

We consider a motion in which the velocity $\mathbf{v}$ and the magnetic field $\mathbf{H}$ are given by

$$\mathbf{v} = (u, v, 0), \quad \mathbf{H} = (H_n, H, 0), \tag{1}$$

where all components are functions of x alone. It then follows from $\operatorname{div} \mathbf{H} = 0$ that

$$H_n = \text{constant}, \tag{2}$$

and from $\mathbf{J} = \operatorname{curl} \mathbf{H}$ that the current lies in the z-direction:

$$\mathbf{J} = (0, 0, J), \quad J = H'(x). \tag{3}$$

If we now assume the conduction equation

$$\mathbf{J} = \sigma(\mathbf{E} + \mu \mathbf{v} \times \mathbf{H}), \tag{4}$$

then the electric field $\mathbf{E}$ also lies in the z-direction (since both $\mathbf{J}$ and $\mathbf{v} \times \mathbf{H}$ do):

$$\mathbf{E} = (0, 0, E_0), \quad E_0 = \text{constant},$$

where the constancy of E_0 follows from $\operatorname{curl} \mathbf{E} = 0$.

Under these conditions the equations of continuity and momentum* reduce to

$$\left.\begin{aligned} &\frac{d}{dx}(\rho u) = 0, \\ &\rho u\frac{du}{dx} = -\frac{dp}{dx}+\mu_1\frac{d^2u}{dx^2}-\mu H\frac{dH}{dx}, \\ &\rho u\frac{dv}{dx} = \mu_2\frac{d^2v}{dx^2}+\mu H_n\frac{dH}{dx}, \end{aligned}\right\} \tag{5}$$

where μ_2 is the coefficient of shear viscosity while $\mu_1-4\mu_2/3$ is the coefficient of bulk viscosity. Similarly, under the assumption that a fluid element receives energy only by heat conduction (coefficient k) and by viscous and electrical dissipation, and assuming the gas to be perfect, we find the specifying equation

$$\begin{aligned} &\rho u\frac{d}{dx}\left[\tfrac{1}{2}(u^2+v^2)+\frac{p}{(\gamma-1)\rho}+\frac{\mu}{2\rho}(H_n^2+H^2)\right] \\ &\quad +\frac{d}{dx}\left[pu-\mu_1 u\frac{du}{dx}-\mu_2 v\frac{dv}{dx}-\frac{\mu}{2}\{(H_n^2-H^2)u+2H_nHv\}-\frac{1}{\sigma}H\frac{dH}{dx}\right] = k\frac{d^2T}{dx^2}, \end{aligned} \tag{6}$$

where γ has its usual meaning.

To these must be added the conduction equation (4) with $\mathbf{J}$ given by (3)

$$\frac{dH}{dx} = \sigma[E_0+\mu(uH-vH_n)]. \tag{7}$$

These five equations (for u, v, p, ρ, H) immediately integrate to give $\rho u = m$ (constant) and

$$\begin{aligned} \mu_1\frac{du}{dx} &= mu+p+\tfrac{1}{2}\mu H^2-A, \\ \mu_2\frac{dv}{dx} &= mv-\mu H_n H-B, \\ k\frac{dT}{dx} &= C+Au+Bv+m\left[\frac{p}{(\gamma-1)\rho}-\tfrac{1}{2}(u^2+v^2)\right]-HE_0-\tfrac{1}{2}\mu H(uH-2H_nv), \\ \frac{1}{\sigma}\frac{dH}{dx} &= E_0+\mu(uH-vH_n), \end{aligned}$$

where A, B, C are constants. Without loss in generality we may take $u > 0$ and hence $m > 0$, and, since we are interested only in flows which tend to uniform conditions ($d/dx = 0$), we also have $A > 0$. When the dimensionless variables

$$U = \frac{m}{A}u, \quad V = \frac{m}{A}v, \quad h = \sqrt{\left(\frac{\mu}{2A}\right)}H, \quad \Theta = \frac{m^2p}{A^2\rho} = \frac{m^2RT}{A^2}$$

* In this paragraph see Goldstein (1957) for the complete equations.

are introduced, the equations become

$$\left.\begin{aligned}\frac{\mu_1}{m}U\frac{dU}{dx} &= -U+\Theta+U^2+Uh^2,\\ \frac{\mu_2}{m}\frac{dV}{dx} &= a+V-2h_n h,\\ \frac{k}{mR}\frac{d\Theta}{dx} &= b+U-aV+\frac{\Theta}{\gamma-1}-e_0 h-\tfrac{1}{2}(U^2+V^2)+2h_n Vh-Uh^2,\\ \frac{m\eta}{A}\frac{dh}{dx} &= \tfrac{1}{2}e_0+Uh-h_n V,\end{aligned}\right\} \tag{8}$$

where $\eta = 1/\mu\sigma$ is the magnetic diffusivity, and

$$a = -\frac{B}{A},\quad b=\frac{mC}{A^2},\quad e_0 = m\sqrt{\left(\frac{2}{\mu A^3}\right)}E_0,\quad h_n = \sqrt{\left(\frac{\mu}{2A}\right)}H_n.$$

Solutions of the system (8) are represented by curves in (U, V, Θ, h)-space on which x is a parameter. In order to visualize their geometry we make the following simplifications. (i) In the third of equations (8) k is set equal to zero; it is known that in the purely gas-dynamical case the neglect of heat conduction has no effect on the general character of shock transition solutions (see Gilbarg 1951). (ii) In the second equation (8) we put $\mu_2 = 0$; this corresponds to the physically unrealistic assumption of zero shear viscosity and non-zero bulk viscosity, but again this is justified in the purely gas-dynamical case (in fact the relevant solution is the same: $V = -a$, whether μ_2 is zero or not). Under these assumptions the system (8) reduces to

$$\left.\begin{aligned}\frac{\mu_1}{m}U\frac{dU}{dx} &= f(U,h) \equiv (\gamma-1)\beta-\gamma U+2(\gamma-1)\alpha h+\frac{\gamma+1}{2}U^2-2(\gamma-1)h_n^2h^2+\gamma Uh^2,\\ \frac{m\eta}{A}\frac{dh}{dx} &= g(U,h)\equiv \alpha-h(2h_n^2-U),\end{aligned}\right\} \tag{9}$$

when V and Θ are eliminated; here

$$\alpha = ah_n+\tfrac{1}{2}e_0,\quad \beta = -(b+\tfrac{1}{2}a^2).$$

2. The singular points

In the (U, h)-plane the infinite isocline of the system (9) is the cubic $f = 0$. The zero isocline is formed by the rectangular hyperbola* $g = 0$ and the h-axis: $U = 0$. We are concerned with integral curves lying in the right half of the plane: $U > 0$, and in particular those joining the singular points given by $f = 0$, $g = 0$. These points are also the intersections of the ellipse

$$f-\gamma hg \equiv \frac{\gamma+1}{2}\left[U-\frac{\gamma}{\gamma+1}\right]^2+2h_n^2\left[h-\frac{(2-\gamma)\alpha}{4h_n^2}\right]^2 + \left[\beta(\gamma-1)-\frac{(2-\gamma)^2\alpha^2}{8h_n^2}-\frac{\gamma^2}{2(\gamma+1)}\right]=0, \tag{10}$$

* For $\alpha = 0$ the results which follow must be understood in an appropriate limit sense.

which is not an isocline, and the rectangular hyperbola (see figure 1). Therefore, there are 0, 2 or 4 real singular points, of which an even number lie on each branch of the hyperbola. On eliminating U between $f = 0$ and $g = 0$, we find that their ordinates are given by the quartic equation

$$F(h) \equiv 2h_n^2 h^4 - (2-\gamma)\alpha h^3 + [2(\gamma+1)h_n^4 - 2\gamma h_n^2 + (\gamma-1)\beta]h^2 - \alpha[2(\gamma+1)h_n^2 - \gamma]h + \tfrac{1}{2}(\gamma+1)\alpha^2 = 0. \quad (11)$$

We now investigate the nature of the singular points, since it is this which determines the way in which a transition flow tends to the corresponding uniform conditions at infinity.

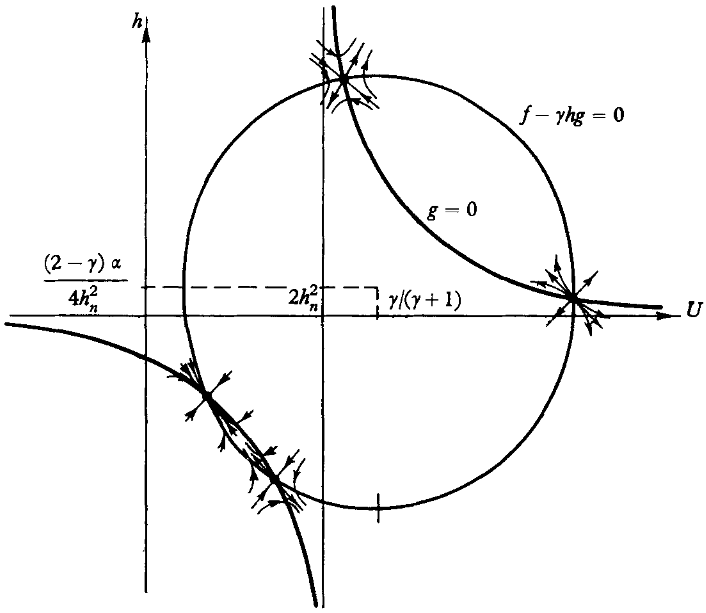

FIGURE 1. Singular points in phase plane.

In order of decreasing $|h|$ the points (in $U>0$) with $h>0$ are alternatively saddle and nodal, and similarly for those with $h<0$. For the characteristic equation of (9) at a singular point (U_0, h_0) is*

$$\kappa^2 - (\bar{f}_U + \bar{g}_h)\kappa + (\bar{f}_U\bar{g}_h - \bar{f}_h\bar{g}_U) = 0, \quad (12)$$

where $\bar{f} = mf/\mu_1 U$, $\bar{g} = Ag/m\eta$ and all values are to be taken at the point, i.e.

$$\bar{f}_U = \frac{m}{\mu_1 U_0}[-\gamma + (\gamma+1)U_0 + \gamma h_0^2], \quad \bar{f}_h = \frac{2m}{\mu_1}h_0 \quad (13a)$$

and
$$\bar{g}_U = \frac{A}{m\eta}h_0, \quad \bar{g}_h = -\frac{A}{m\eta}\frac{\alpha}{h_0}. \quad (13b)$$

The discriminant of this equation is

$$(\bar{f}_U - \bar{g}_h)^2 + 4\bar{f}_h\bar{g}_U,$$

* This equation determines the exponential solutions ($e^{\kappa x}$) of (9) when the right-hand sides are linearized in the neighbourhood of the point (cf. § 5).

which is positive since $\bar{f}_h$ and $\bar{g}_U$ have the same sign; hence the roots of (12) are real. Also the constant term has the same sign as

$$
\begin{aligned}
&-\frac{1}{U_0 h_0}[\alpha\{-\gamma+(\gamma+1)U_0+\gamma h_0^2\}+2U_0 h_0^3] \\
&= -\frac{1}{U_0 h_0^2}[4h_n^2 h_0^4-(2-\gamma)\alpha h_0^3+\alpha\{2(\gamma+1)h_n^2-\gamma\}h_0-(\gamma+1)\alpha^2] \\
&= -\frac{h_0}{U_0}\left[\frac{d}{dh}\left(\frac{F(h)}{h^2}\right)\right]_{h=h_0} \qquad (14)
\end{aligned}
$$

where $F(h)$ is given in (11). Now $hd[F(h)/h^2]/dh$ is clearly positive for sufficiently large $|h|$ and has opposite signs at consecutive zero of $F(h)/h^2$ on the same side of the U-axis. Since a negative constant term implies a saddle point, and a positive one a nodal point, the result follows.

Since there may be two nodal points in the region of interest the possibility arises of there being an infinity of integral curves joining them, i.e. a family of transitions between given states at $x = \pm\infty$. We shall discuss two particular cases; in the first this does not occur, in the second it does.

3. The case $H_n = 0$

This was discussed first by Marshall (1955) and then by Burgers (1957). The former did not completely discuss the limit of vanishing viscosity; the latter assumed zero viscosity from the start and hence had to introduce extraneous assumptions.

We return for a moment to the original equations (8), with μ_2 not necessarily zero, and set $h_n = 0$. Then the second of these can be integrated explicitly, the only relevant solution being $V = -a$ (all others are exponentially large at $x = +\infty$). But this is precisely what is used to eliminate V and obtain equations (9) when $\mu_2 = 0$. Hence, in the present case our discussion is in fact independent of the assumption $\mu_2 = 0$.

The rectangular hyperbola becomes

$$g \equiv \alpha + hU = 0,$$

and the ellipse (10) degenerates into the parabola

$$f-\gamma hg \equiv \frac{\gamma+1}{2}\left[U-\frac{\gamma}{\gamma+1}\right]^2-(2-\gamma)\alpha h+\left[(\gamma-1)\beta-\frac{\gamma^2}{2(\gamma+1)}\right] = 0.$$

Since the system (9) is unaltered on replacing α, h by $-\alpha, -h$ we may take $\alpha \leqslant 0$. Then these curves will lie as in figure 2, always having an intersection 3 at which $U < 0$, and, for suitable values of α, β, another two intersections 1, 2 at which U is always positive. This can be read off from (11), which also shows that the fourth intersection lies at infinity on the lower branch of the hyperbola.

Only the points 1 and 2 are of physical interest. Between them f is negative $(=f-\gamma hg)$ on the hyperbola and positive $(=\gamma hg)$ on the parabola. Hence the infinite isocline: $f = 0$ lies above the hyperbola and below the parabola. Since each horizontal cuts it in at most two points (see (9)), it has just one extremum (a maximum) between 1 and 2 if it has positive slope at 2 and none if it has negative slope. From (13*a*) the condition for a maximum is therefore $U_2^2-\gamma U_2(1-U_2-h_2^2) = U_2^2-\gamma\Theta_2 < 0$, or in the original variables: $u_2^2 < \gamma p_2/\rho_2 = a_2^2$. Within the region R bounded by the arcs 1 2 of $f = 0$ and $g = 0$ the integral curves have negative slope, with x increasing in the negative U-direction on each.

We saw in the last section that 2 is a saddle point and 1 is a nodal point. The singular directions at either point are determined by

$$Q(m) \equiv \bar{f}_h m^2 + (\bar{f}_U - \bar{g}_h)\, m - \bar{g}_U = 0;$$

hence, since $Q(0) < 0$ and $Q(\pm\infty) > 0$ (see (13)), one has positive slope and the other negative. It now follows, as in the purely gas-dynamical case (see Gilbarg 1951), that there is a unique transition solution joining the points 1 and 2, which

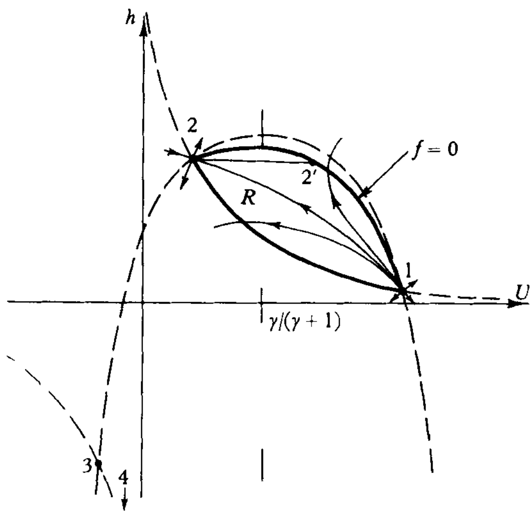

FIGURE 2. Transition curve for $H_n = 0$.

correspond to $x = -\infty$ and $x = +\infty$, respectively,* and this curve lies in R. Moreover, when the infinite isocline is monotonic between 1 and 2, the limiting transition as $\mu_1 \to 0$ follows this isocline; whereas when there is a maximum, it follows the isocline until attaining (at 2′) the same ordinate as 2 and then the horizontal into 2. This second behaviour is due to the fact that no integral curve can attain positive slope in R.

The straight segment 2 2′ corresponds to a purely gas-dynamical shock (x constant in limit); across it the Rankine–Hugoniot conditions hold. For $f = 0$ is the result of eliminating V and Θ between the equations obtained by setting the right sides (with $h_n = 0$) of the first three equations (8) equal to zero. Hence

$$U_2+\frac{\Theta_2}{U_2} = 1-h_2^2 = 1-h_{2'}^2 = U_{2'}+\frac{\Theta_{2'}}{U_{2'}},$$

$$\tfrac{1}{2}U_2^2+\frac{\gamma}{\gamma-1}\Theta_2 = -b+aV_2+e_0h_2+\tfrac{1}{2}V_2^2 = -b+aV_{2'}+e_0h_{2'}+\tfrac{1}{2}V_{2'}^2$$

$$= \tfrac{1}{2}U_{2'}^2+\frac{\gamma}{\gamma-1}\Theta_{2'},$$

* The sign of (14) expresses that u^2 is greater than $\gamma p/\rho+\mu H^2/\rho$ at 1 and less at 2, i.e. the transition flow is initially supersonic and finally subsonic.

which, combined with $\rho_2 u_2 = \rho_{2'} u_{2'}$ and expressed in the original variables, are precisely the Rankine–Hugoniot conditions. The point 2 also corresponds to the whole uniform limit state behind the shock.

Thus, we see that a transition flow in an electrically conducting fluid of vanishingly small viscosity contains a gas-dynamical shock if and only if the final state is subsonic (in the gas-dynamical sense), and that when it does the whole adjustment region (to hydromagnetic values) lies upstream of the shock. The magnetic field is continuous across the shock, though dH/dx undergoes a jump $\sigma(E_0 + \mu u_{2'} H_2)$.

4. The case $H = 0$ in initial state

One possibility is that $H \equiv 0$. The transition is then a purely fluid-dynamical one with no interaction between flow and magnetic field. We shall see immediately, however, that this is not the only possibility under certain circumstances; indeed, our assumption only implies $\alpha = 0$ in (9).

Clearly the present section covers Friedrichs' switch-on shock discontinuities ($H = 0$ initially but not finally); it also covers his switch-off ($H = 0$ finally but not initially), and transverse (H changes sign but not magnitude) shock discontinuities, since either assumption concerning the end-points leads to $\alpha = 0$ also.

The zero isocline degenerates into the pair of straight lines

$$U = 2h_n^2, \quad h = 0,$$

while the infinite isocline

$$\left[U - \frac{2(\gamma-1)}{\gamma} h_n^2\right]\left[\frac{\gamma+1}{2} U + \gamma h^2 + \frac{\gamma^2-1}{\gamma} h_n^2 - \gamma\right] = 2(\gamma-1) h_n^2 \left[1 - \frac{\gamma^2-1}{\gamma^2} h_n^2\right] - (\gamma-1)\beta$$

represents a hyperbola in the (U, h^2)-plane. [We use this rather than the (U, h)-plane in the present section.] The three points of intersection are

$$1, 2: \quad U = \frac{\gamma}{\gamma+1}(1 \pm k), \quad h^2 = 0,$$

$$3: \quad U = 2h_n^2, \quad h^2 = \frac{\gamma^2}{4(\gamma+1) h_n^2}\left[k^2 - \left(1 - \frac{2\overline{\gamma+1}\, h_n^2}{\gamma}\right)^2\right],$$

where $k = \sqrt{\{1 - 2(\gamma^2-1)\beta/\gamma^2\}}$ (which must be real, by assumption).

Thus, there are five ranges of the parameter h_n to consider:*

(i) $$2h_n^2 < \frac{\gamma}{\gamma+1}(1-k) = U_2,$$

(ii$a$) $$\frac{\gamma}{\gamma+1}(1-k) < 2h_n^2 < \frac{\gamma^2}{\gamma^2-1}(1-k),$$

(ii$b$) $$\frac{\gamma^2}{\gamma^2-1}(1-k) < 2h_n^2 < \frac{\gamma}{\gamma+1}(1+k) = U_1,$$

* From the first of equations (8), $\Theta_1 = U_1(1-U_1)$. Hence, with positive temperature at 1, k is less than $1/\gamma$.

(iii$a$) $$\frac{\gamma}{\gamma+1}(1+k) < 2h_n^2 < \frac{\gamma^2}{\gamma^2-1}(1+k),$$

(iii$b$) $$2h_n^2 > \frac{\gamma^2}{\gamma^2-1}(1+k).$$

The different situations are illustrated in figures 3–6. In case (i) (figure 3), the point 3 is discarded and 1, 2 lie on the same branch of the hyperbola with 1 a nodal and 2 a saddle point. For (ii), 3 is a saddle point, while the points 1 and 2 are nodal, lying on the same branch when (ii$a$) holds (figure 4) and on different branches when (ii$b$) holds (figure 5). With (iii) (figure 6), the point 3 is again discarded, the point 1 is a saddle, and point 2 a node, and the latter lie on different or the same branches of the hyperbola according as (iii$a$) or (iii$b$) holds. The integral curves are easily sketched once the singular directions at 1, 2 and 3 have been determined and the signs of dU/dx and dh/dx in the various regions found.

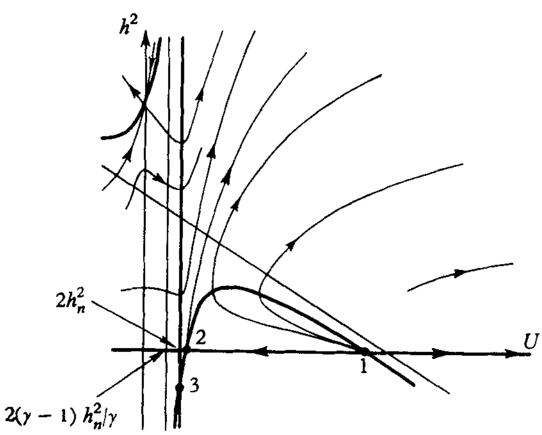

FIGURE 3. Integral curves for $\alpha = 0$; case (i).

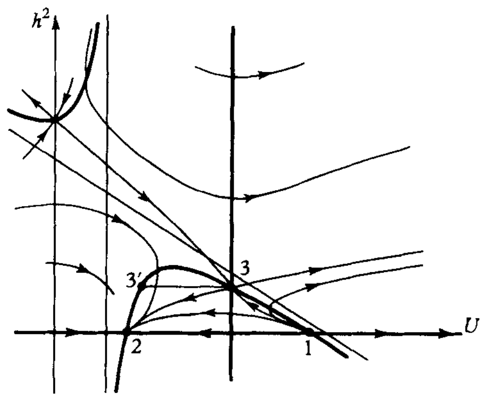

FIGURE 4. Integral curves for $\alpha = 0$; case (ii$a$).

When (i) or either of (iii) holds there is a unique transition: it starts at 1 and ends at 2, and has $H = 0$ throughout, so that there is no interaction between the fluid motion and the magnetic field. The end-points 1 and 2 satisfy the ordinary Rankine–Hugoniot shock conditions. The same transition can occur in case (ii), but now there is not only a second end-point* 3 (to which the transition is unique), but also a whole family of possible transitions from 1 to 2. Within each, H^2 increases from zero to a maximum and then decreases to zero again.

Before discussing this lack of uniqueness in greater detail we make one further observation. If the initial state and magnetic field are chosen so as to give the point 3 in figure 4, there will be a unique transition ending at the point† 2. Now the hyperbola has its maximum‡ to the left of 3, so that as $\mu_1 \to 0$ this transition

* The pair 1, 3 represents the switch-on shock, while the two images of 3 in the (U, h)-plane represent the transverse shock. There is clearly no integral curve joining the latter.

† The pair 3, 2 in figures 4, 5 represents the switch-off shock.

‡ For smaller h_n the maximum lies on the right side; the change takes place as u_3 increases through $a_3 = \sqrt{(\gamma p_3/\rho_3)}$.

will tend to follow the horizontal 3 3′ and then the infinite isocline to 2. A proof quite similar to that at the end of § 3 shows that 3 and 3′ satisfy the ordinary gas-dynamical shock conditions. However, in contrast to § 3 the whole adjustment region now lies downstream of the shock.

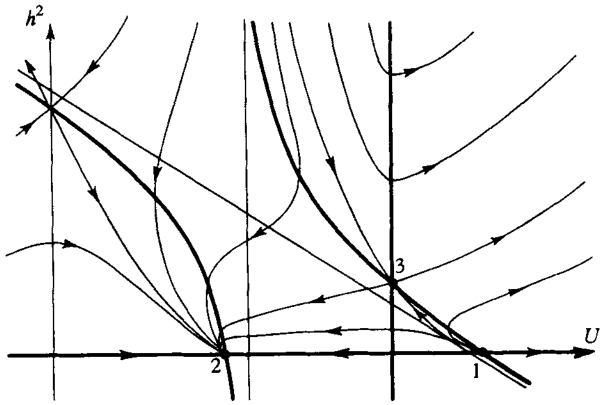

FIGURE 5. Integral curves for $\alpha = 0$; case (ii$b$).

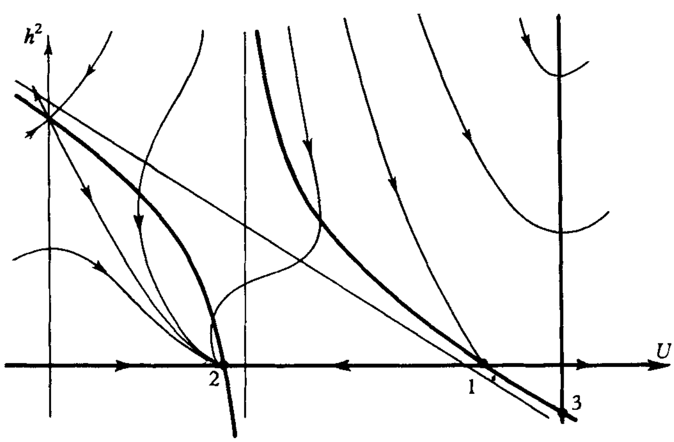

FIGURE 6. Integral curves for $\alpha = 0$; case (iii$a$). In case (iii$b$) the hyperbola has the same form as in figures 3 and 4, the points 1 and 2 now lying on the upper branch.

5. Axially symmetric flow

A simple way to close the current lines is to consider a flow in which the velocity **v** and magnetic field **H** are given by (1) in a cylindrical co-ordinate system (x, r, θ), where now all components are functions of x and r but not of θ. Then (2) is replaced by

$$\frac{\partial H_n}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rH) = 0, \tag{15}$$

and the currents form closed loops (circles) in the fluid with

$$\mathbf{J} = (0, 0, J), \quad J = \frac{\partial H}{\partial x} - \frac{\partial H_n}{\partial r}.$$

From (4) we then see that $\mathbf{E}$ has only a θ-component:

$$\mathbf{E} = \left(0, 0, \frac{E_0}{r}\right), \quad E_0 = \text{constant}, \tag{16}$$

where curl $\mathbf{E} = 0$ has been used to obtain the form of this component.

We now consider the ways in which a uniform stream: $u = u_0$, $v = 0$, $p = p_0$, $\rho = \rho_0$ with $H = 0$ and $H_n = H_{n0}$, at $x = -\infty$ can be disturbed. In the one-dimensional case such disturbances are represented by the integral curves leaving a point such as 1 in figures 3–6. Here we discuss only the immediate behaviour and hence replace all coefficients in the non-linear equations of continuity, momentum, energy balance, and electric conduction by their zero values. Using u, v, p, ρ, H and H_n for their own disturbances now, we find (5), (6) and (7) replaced by

$$\rho_0 \frac{\partial u}{\partial x} + u_0 \frac{\partial \rho}{\partial x} + \frac{\rho_0}{r}\frac{\partial}{\partial r}(rv) = 0, \tag{17a}$$

$$\rho_0 u_0 \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \mu_1 \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rv)\right], \tag{17b}$$

$$\rho_0 u_0 \frac{\partial v}{\partial x} = -\frac{\partial p}{\partial r} + \mu_1 \frac{\partial}{\partial r}\left[\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rv)\right] + \mu H_{n0}\left(\frac{\partial H}{\partial x} - \frac{\partial H_n}{\partial r}\right), \tag{17c}$$

$$\frac{\partial p}{\partial x} - a_0^2 \frac{\partial \rho}{\partial x} = 0, \quad a_0^2 = \gamma \frac{p_0}{\rho_0}, \tag{17d}$$

$$\eta\left[\frac{\partial H}{\partial x} - \frac{\partial H_n}{\partial r}\right] = u_0 H - H_{n0} v; \tag{17e}$$

in obtaining this last equation from (4) we must set $E_0 = 0$ in (16) to ensure that $\partial H/\partial x \to 0$ as $x \to -\infty$.

The six functions u, v, p, ρ, H_n, H must therefore satisfy equations (15) and (17). For reference we first determine the corresponding solution in the one-dimensional case, which is obtained by replacing all $\partial/\partial r$ terms by zero. The resulting homogeneous system is solved by setting all variables proportional to $e^{\lambda x}$, where either $\lambda = 0$ (which we exclude) or $H_n = 0$ and*

$$\lambda = \frac{\rho_0 u_0}{\mu_1}\left(1 - \frac{a_0^2}{u_0^2}\right) = \lambda_1 \quad \text{or} \quad \lambda = \frac{u_0}{\eta}\left(1 - \frac{\mu H_{n0}^2}{\rho_0 u_0^2}\right) = \lambda_2.$$

The complete (relevant) solution is

$$p = a_0^2 \rho = -\frac{\rho_0 a_0^2}{u_0} u, \quad v = \frac{\mu H_{n0}}{\rho_0 u_0} H, \quad H_n = 0,$$
$$u = A\, e^{\lambda_1 x}, \quad H = B\, e^{\lambda_2 x}, \tag{18}$$

where A and B are arbitrary constants. Each ratio $A{:}B$ determines a different disturbance of the initially uniform stream, the remaining scale factor corresponding to choice of the origin of x.

* At the point 1 in figures 4 and 5, λ_1 and λ_2 are both positive of course. At the point 3 they are both negative, confirming that the latter is approached as $x \to +\infty$.

We now set all variables proportional to $e^{\lambda x}$ in (15) and (17), denoting the factor in each case (a function of r) by the variable itself. Then from (15), (17 a), (17 d) and (17 e) we find

$$\left.\begin{aligned} p = a_0^2\rho &= -\frac{\rho_0 a_0^2}{u_0}\left[u+\frac{1}{\lambda}P\left\{-\frac{\eta}{\lambda H_{n0}}\frac{\partial}{\partial r}P+\frac{1}{H_{n0}}(u_0-\lambda\eta)\right\}H\right], \\ v &= \left[-\frac{\eta}{\lambda H_{n0}}\frac{\partial}{\partial r}P+\frac{1}{H_{n0}}(u_0-\lambda\eta)\right]H, \quad H_n = -\frac{1}{\lambda}PH, \end{aligned}\right\} \tag{19}$$

where $P \equiv \dfrac{1}{r}\dfrac{\partial}{\partial r}r$ and from (17 b), (17 c) u, H must satisfy

$$A_1 u+\left(C_1+D_1 P\frac{\partial}{\partial r}\right)PH = 0, \tag{20a}$$

$$A_2\frac{\partial u}{\partial r}+\left(B_2+C_2\frac{\partial}{\partial r}P+D_2\frac{\partial}{\partial r}P\frac{\partial}{\partial r}P\right)H = 0, \tag{20b}$$

with

$$A_1 = \rho_0 u_0-K, \quad C_1 = -\frac{KL}{\lambda H_{n0}}, \quad D_1 = \frac{\eta K}{\lambda^2 H_{n0}},$$

$$A_2 = -K, \quad B_2 = \frac{\lambda}{H_{n0}}(\rho_0 u_0 K-\mu H_{n0}^2), \quad C_2 = -\frac{1}{\lambda H_{n0}}[KL+\rho_0 u_0(u_0-L)+\mu H_{n0}^2],$$

$$D_2 = \frac{\eta K}{\lambda^2 H_{n0}}, \quad K = \frac{\rho_0 a_0^2}{u_0}+\mu_1\lambda, \quad L = u_0-\eta\lambda.$$

When $\lambda = \lambda_1$ the coefficient A_1 is zero and (20 a) becomes

$$\left(C_1+D_1 P\frac{\partial}{\partial r}\right)PH = P\left(C_1+D_1\frac{\partial}{\partial r}P\right)H = 0,$$

so that

$$H = \frac{A}{r}+\mathscr{C}_1(mr), \quad m = \sqrt{\frac{C_1}{D_1}}, \tag{21}$$

where $\mathscr{C}_1$ is the general solution of Bessel's equation of order one, and A is an arbitrary constant. Pending further discussion, we take $\mathscr{C}_1 = 0$; then from (20 b)

$$u = -\frac{B_2}{A_2}A\log r+B.$$

Similarly, for $\lambda = \lambda_2 : B_2 = 0$ and

$$H = Cr+\frac{D}{r}, \quad u = -\frac{2D_1}{A_1}C,$$

where C and D are arbitrary constants and zero has been taken as the solution of a Bessel equation.

The general disturbance is therefore given by (19) and

$$u = \left[-\frac{B_2}{A_2}A\log r+B\right]e^{\lambda_1 x}-\frac{2D_1}{A_1}C\,e^{\lambda_2 x},$$

$$H = \frac{A}{r}e^{\lambda_1 x}+\left[Cr+\frac{D}{r}\right]e^{\lambda_2 x}.$$

Thus for $r \to \infty$, (i) with $s = x+(\log r)/\lambda_2$ fixed, we find

$$u = 0, \quad H = C\,e^{\lambda_2 s};$$

(ii) with $C = 0$ and $s = x + (\log\log r)/\lambda_1$ fixed

$$u = -\frac{B_2}{A_2} A\, e^{\lambda_1 s}, \quad H = 0;$$

(iii) with $C = A = 0$ and $s = x$ fixed

$$u = B\, e^{\lambda_1 s}, \quad H = 0;$$

and, finally, (iv) with $C = A = B = 0$ and $s = x - (\log r)/\lambda_1$ fixed

$$u = 0, \quad H = D\, e^{\lambda_2 s}.$$

We conclude that the only one-dimensional disturbances which are the limits as $r \to \infty$ of axially symmetric ones are those corresponding to $A = 0$ or $B = 0$ in (18). This singles out what may be called the two principal solutions of (9) through the point 1 in figures 4 and 5, and it is clear that one of the principal solutions is the purely fluid-dynamical transition $H = 0$. The other may be ignored, at least for small enough η/μ_1.*

The reason for excluding the Bessel function in (21) is now clear. For large r, the corresponding solution would either oscillate, and hence not have a one-dimensional form, or else behave exponentially (small or large): $e^{\tau r}/r^{\frac{1}{2}}$, $\tau = \pm im$. For the latter we would have to take $s = x + \tau r - (\log r)/2\lambda_1$, which corresponds to a shock front at an oblique angle to the incident stream.

Final remarks

The lack of uniqueness found in §4 cannot be attributed to the unrealistic assumption $\mu_2 = 0$; without this assumption there is still an infinite number of integral curves in (U, V, h)-space joining the points 1 and 2 in figures 4 and 5, which are now a nodal and nodal-saddle point, respectively. Also the discussion in §5 need not be restricted in this way and the (three) principal solutions through 1 are again singled out.†

Moreover, the same results are obtained in the limit $\mu_1, \mu_2 \to 0$ as for our special case $\mu_2 = 0$, $\mu_1 \to 0$. For a limiting transition curve in (U, V, h)-space clearly follows the curve obtained by setting the right sides of the first three equations (8) equal to zero, except possibly for a segment lying in a plane $h =$ constant whose ends lie on this curve (cf. end of §3). At these ends the value of $V = -a + 2h_n h$ is the same, so that V is constant along the segment; otherwise dV/dx would have to change sign on transition curves at points outside the plane

$$V - 2h_n h + a = 0. \tag{22}$$

Thus in the limit $\mu_1, \mu_2 \to 0$ the transition curve lies in the plane (22) and follows the same path as for $\mu_1 \to 0$, $\mu_2 = 0$.

* Its slope at 1 is $dh^2/dU = [U_1(U_1 - 2h_n^2) - \frac{1}{2}(\gamma+1)\,\epsilon(U_1 - U_2)]/\epsilon[\gamma U_1 - 2(\gamma-1)\,h_n^2]$, where $\epsilon = m^2\eta/A\mu_1$. Thus for $\epsilon < 2U_1(U_1 - 2h_n^2)/(\gamma+1)\,(U_1 - U_2) = \epsilon_0$ the curve leaves 1 with positive slope and joins 1 to 2 below the U-axis (figures 4 and 5 were drawn for ϵ large); such a join must be rejected (h is imaginary). As ϵ increases past ϵ_0 we may expect the curve to continue to join 1 and 2, but now above the U-axis. It would then provide an alternative transition to $H \equiv 0$.

† And similarly at 2, where one of the solutions may be discarded since the point is approached as $x \to -\infty$.

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